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A Linear Asymptotic Theory for Anisotropic Shells

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SUMMARY

In the standard treatments for anisotropic shells based on the Kirchoff hypothesis, it is necessary to make certain restrictions on the type of anisotropy and the resulting theories involve only six elastic constants.

In the present work, a shell theory is obtained by an "asymptotic" or "perturbation" method which does not require any restriction on the anisotropy.

It is found that, in those cases in which the extensional and bending strains are of the same order of magnitude, the leading terms satisfy the classical equations and depend only on the same six elastic constants.

It is seen however that in some cases the full anisotropy is significant and it is shown that in the extension of a plate the anisotropy can produce displacements normal to the plate.

Introduction

In the standard treatments of anisotropic shells (e.g. Ambartsumyan [1]) the assumption is made that the mid-surface of the shell is a surface of elastic symmetry. This assumption then leads to theories which involve only six of the twenty-one distinct components of the elasticity tensor.

In recent work by Widera and Johnson [2, 3] an asymptotic method is used to develop fully anisotropic non homogeneous dynamic theories for plates and for cylindrical shells. One important conclusion of this work is that even without any assumption of elastic symmetry, the governing equations still contain only the same six components of the elasticity tensor as the standard treatments. Thus the assumption of elastic symmetry is not necessary in these cases.

In the present work, which is an asymptotic investigation of the statics of a general anisotropic shell, the same conclusion: that an assumption of elastic symmetry is not necessary, is obtained in cases where the extensional and bending strains are of the same order of magnitude.

In section 3, however, cases in which the extensional and bending strains are of different orders of magnitude are examined and it is found that in these cases, full anisotropy may play a significant role. The equilibrium equations obtained in section 3 are believed to be new and they indicate that:

(1) Extension in an anisotropic plate or shallow shell may be accompanied by bending of a smaller magnitude.

(2) Bending in an anisotropic plate or shell may be accompanied by extension of a smaller magnitude.

The present work also indicates that the error in an anisotropic theory will generally be larger than that in a corresponding isotropic theory.

In their papers [2, 3] Johnson and Widera develop the theory in terms of the displacements making assumptions on the order of magnitude of the various components of the displacement vector. Johnson and Widera do not investigate all possibilities since they are concerned with specific cases. Because the present work deals with statics, it is not necessary to use the displacements and the theory is developed from the point of view of the strains. The disadvantage of using the strains is that compatability equations must be included in the theory but it is felt that this is outweighed by the fact that the scaling of the strains is much easier than that of the displacements and that the extensional and bending strains are easily identified, so that different orders of magnitude for these strains may be considered.

A further advantage is that an extension to a nonlinear small strain case is made much simpler by the use of the strains rather than the displacements (e.g. Westbrook [4]). The relationship between the displacements and the strains is given at the end of section 3.

1. Coordinates and Scaling

It is assumed that the strains ε_{ij} are everywhere of magnitude ε or less.

Let x_1, x_2 be curvilinear coordinates in the mid-surface of the shell and x_3 be the coordinate perpendicular to the mid-surface. The surfaces of the shell are then given by $x_3 = \pm h$. The coordinates are scaled as follows: let $\xi_\alpha = x_\alpha/L$ ($\alpha = 1, 2$) where L is a typical length which depends on the boundary data and shell geometry and let $\zeta = x_3/h$. The curvature tensor $B_{\alpha\beta}$ of the mid surface is scaled by a length R to give the scaled curvature tensor $K_{\alpha\beta} = RB_{\alpha\beta}$ and a scaled surface Christoffel symbol $\Gamma^\alpha_{\beta\gamma}$ is defined by

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} R A^{\alpha\delta}(A_{\alpha\beta,\gamma} + A_{\delta\gamma,\beta} - A_{\beta\gamma,\delta}) .$$

(Here and in what follows, the comma denotes partial differentiation.)

If the length R is determined in the manner of John [5], the dimensionless quantities $K_{\alpha\beta}, \Gamma^\alpha_{\beta\gamma}$ are $O(1)$.

If the constitutive equations are

$$\varepsilon_{ij} = G_{ijkl} t^{kl} ,$$

then the following scaled dimensionless elasticity tensor and stress tensor are defined

$$\mu_{ijkl} = G G_{ijkl} , \quad \tau^{kl} = G^{-1} t^{kl}$$

where τ^{kl} are of the order ε or less and μ_{ijkl} are of order unity or less.

The following small parameters are defined

$$\delta = \frac{h}{L}, \quad \theta = \frac{h}{R} \text{ and } \gamma = \operatorname{Max}(\delta, \theta^{\frac{1}{2}}) .$$

For the present linear theory to be valid it is required that $\varepsilon \leqq \gamma^4$ (see [4]). In what follows no assumptions are made about the relative magnitudes of δ and θ beyond the assumption $\delta \geqq \theta$ ($R \geqq L$). This means that in some cases terms appearing in our equations would be as small as terms already omitted and could therefore also be omitted. It is felt however, that since only some leading terms are considered, the retention of more terms is preferable to the division into separate cases which is necessary if assumptions on the relative magnitudes of θ and δ are made.

2. Shell Equations

The equations of the three-dimensional theory are: the equilibrium equations

$$t^{ij}{}_{;j} = 0 ,$$

the compatability equations

$$\varepsilon_{ij;hk} + \varepsilon_{hk;ij} - \varepsilon_{ih;kj} - \varepsilon_{kj;hi} = 0 ,$$

and the constitutive equations

$$\varepsilon_{ij} = G_{ijkl} t^{kl} .$$

In these equations ";" denotes covariant differentiation. The equations of the shell theory are derived from these equations together with the boundary conditions, $t^{i3} = \frac{1}{2} G(Q^i \pm P^i)$ on $x_3 = \pm h$.

In what follows the following common conventions are used; Greek letters will take the values 1 and 2, latin letters the values 1, 2 and 3, "|" will denote scaled covariant differentiation with respect to the mid-surface metric.

After scaling the equilibrium equations become,

$$\tau^{\alpha 3}{}_{,\zeta} + \delta\tau^{\alpha\beta}|_{\beta} - 2\theta K^{\alpha}_{\beta}\tau^{3\beta} - \theta K^{\beta}_{\beta}\tau^{\alpha 3} = O(\varepsilon\gamma^4) \tag{1}$$

$$\tau^{33}{}_{,\zeta} + \delta\tau^{\alpha 3}|_{\alpha} + \theta K_{\alpha\beta}\tau^{\alpha\beta} - \theta K^{\alpha}_{\alpha}\tau^{33} = O(\varepsilon\gamma^4) \tag{2}$$

and the boundary conditions are

$$\tau^{i3} = \tfrac{1}{2}(Q^i \pm P^i) \quad \text{on} \quad \zeta = \pm 1\,. \tag{3}$$

The constitutive equations may be written

$$\varepsilon_{\alpha\beta} = \mu_{\alpha\beta\gamma\delta}\tau^{\gamma\delta} + 2\mu_{\alpha\beta\gamma 3}\tau^{\gamma 3} + \mu_{\alpha\beta 33}\tau^{33} \tag{4}$$

$$\varepsilon_{\alpha 3} = \mu_{\alpha 3\gamma\delta}\tau^{\gamma\delta} + 2\mu_{\alpha 3\gamma 3}\tau^{\gamma 3} + \mu_{\alpha 333}\tau^{33} \tag{5}$$

$$\varepsilon_{33} = \mu_{33\gamma\delta}\tau^{\gamma\delta} + 2\mu_{33\gamma 3}\tau^{\gamma 3} + \mu_{3333}\tau^{33} \tag{6}$$

and the elasticity tensor μ_{ijkl} must satisfy the following symmetry conditions,

$$\mu_{\alpha\beta\gamma\delta} = \mu_{\beta\alpha\gamma\delta} = \mu_{\alpha\beta\delta\gamma} = \mu_{\gamma\delta\alpha\beta}$$

$$\mu_{\alpha\beta\gamma 3} = \mu_{\beta\alpha\gamma 3} = \mu_{\alpha\beta 3\gamma} = \mu_{\beta\alpha 3\gamma} = \mu_{\gamma 3\alpha\beta} = \mu_{3\gamma\alpha\beta} = \mu_{3\gamma\beta\alpha}$$

$$\mu_{\alpha 333} = \mu_{3\alpha 33} = \mu_{33\alpha 3} = \mu_{33 3\alpha} \dots$$

Thus μ_{ijkl} has at most 21 distinct components.

When scaled, the compatability equations, with $i=\alpha$, $j=\beta$, $h=k=3$, give

$$\varepsilon_{\alpha\beta,\zeta\zeta} - \delta(\varepsilon_{\alpha 3}|_{\beta} + \varepsilon_{\beta 3}|_{\alpha})_{,\zeta} = O(\varepsilon\gamma^2)\,.$$

Therefore

$$\varepsilon_{\alpha\beta} = E_{\alpha\beta} + \zeta W_{\alpha\beta} + \delta\int_0^{\zeta}(\varepsilon_{\alpha 3}|_{\beta} + \varepsilon_{\beta 3}|_{\alpha})\,d\zeta + O(\varepsilon\gamma^2) \tag{7}$$

where $E_{\alpha\beta}$, $W_{\alpha\beta}$ which are independent of ζ, may be identified with the extensional and bending strains.

If use is made of equation (7) the remaining compatability equations give

$$e^{\alpha\beta}e^{\sigma\tau}[\delta^2 E_{\alpha\tau}|_{\sigma\beta} - \theta K_{\alpha\tau}W_{\sigma\beta}] = O(\varepsilon\gamma^4) \tag{8}$$

$$e^{\alpha\beta}W_{\alpha\gamma}|_{\beta} = O(\varepsilon\gamma^2) \tag{9}$$

where $e^{\alpha\beta}$ is the two-dimensional permutation tensor. These equations are basic equations of any linear shell theory and are the Gauss–Codazzi equations of the deformed middle surface (cf. Koiter [6]). These are basic equations in our theory.

From the equations of equilibrium (1), (2) and the boundary condition (3), it is seen that

$$\tau^{i3} = Q^i + O(\varepsilon\delta)\,, \qquad P^i = O(\varepsilon\delta)\,.$$

In almost all problems of practical interest, it is assumed that Q^i and P^i are of the same order of magnitude. If this is done here

$$\tau^{i3} = O(\varepsilon\delta)\,, \quad Q^i = O(\varepsilon\delta)\,, \quad P^i = O(\varepsilon\delta)\,.$$

If $\tau^{\alpha 3} = O(\varepsilon\delta)$ is used in (2) it is seen that

$$\tau^{33} = O(\varepsilon\gamma^2)\,, \quad Q^3 = O(\varepsilon\gamma^2)\,, \quad P^3 = O(\varepsilon\gamma^2)\,.$$

The constitutive equations (4), (5) now give

$$E_{\alpha\beta} + \zeta W_{\alpha\beta} = \mu_{\alpha\beta\gamma\delta}\tau^{\gamma\delta} + O(\varepsilon\delta)\,, \quad \varepsilon_{\alpha 3} = \mu_{\alpha 3\gamma\delta}\tau^{\gamma\delta} + O(\varepsilon\delta)\,.$$

At this point the following assumptions on the tensor μ_{ijkl} is made. The tensor μ_{ijkl} may be expanded in powers of ζ and has the following form,

$$\mu_{ijkl} = \sum_{s=0}^{\infty} \overset{s}{\mu}_{ijkl} \gamma^{2s} \zeta^s$$

where $\overset{s}{\mu}_{ijkl}$ are independent of ζ and all are of order one or less.

If $\overset{0}{v}^{\alpha\beta\gamma\delta}$ is a tensor with the same symmetries as $\overset{0}{\mu}_{\alpha\beta\gamma\delta}$ and such that

$$\overset{0}{v}^{\alpha\beta\gamma\delta} \overset{0}{\mu}_{\gamma\delta\sigma\tau} = \tfrac{1}{2}(\delta^\alpha_\sigma \delta^\beta_\tau + \delta^\alpha_\tau \delta^\beta_\sigma)$$

then

$$\tau^{\alpha\beta} = \overset{0}{v}^{\alpha\beta\gamma\delta} E_{\gamma\delta} + \zeta \overset{0}{v}^{\alpha\beta\gamma\delta} W_{\gamma\delta} + O(\varepsilon\delta) \tag{10a}$$

$$= N^{\alpha\beta} + \zeta M^{\alpha\beta} + O(\varepsilon\delta) \quad \text{say}. \tag{10b}$$

The substitution of these expressions into the equilibrium equations (1), (2) gives

$$\tau^{\alpha 3}{}_{,\zeta} + \delta N^{\alpha\beta}|_\beta + \delta\zeta M^{\alpha\beta}|_\beta = O(\varepsilon\delta^2 \text{ or } \varepsilon\gamma^2\delta) = O(\mathrm{Max}(\varepsilon\delta^2, \varepsilon\gamma^2\delta))$$

$$\tau^{33}{}_{,\zeta} + \delta\tau^{\alpha 3}|_\alpha + \theta K_{\alpha\beta} N^{\alpha\beta} + \theta\zeta K_{\alpha\beta} M^{\alpha\beta} = O(\varepsilon\gamma^4).$$

These equations are integrated with respect to ζ and the boundary conditions are then used to produce the following shell equations and expressions for τ^{i3},

$$\delta N^{\alpha\beta}|_\beta + \tfrac{1}{2} P^\alpha = O(\varepsilon\delta^2 \text{ or } \varepsilon\gamma^2\delta) \tag{11a}$$

$$\tfrac{1}{3}\delta^2 M^{\alpha\beta}|_{\alpha\beta} + \theta K_{\alpha\beta} N^{\alpha\beta} + \tfrac{1}{2} P^3 + \tfrac{1}{2}\delta Q^\alpha|_\alpha = O(\varepsilon\gamma^2\delta) \tag{12a}$$

or

$$\delta(\overset{0}{v}^{\alpha\beta\gamma\delta} E_{\gamma\delta})|_\beta + \tfrac{1}{2} P^\alpha = O(\varepsilon\delta^2 \text{ or } \varepsilon\gamma^2\delta) \tag{11b}$$

$$\tfrac{1}{3}\delta^2 (\overset{0}{v}^{\alpha\beta\gamma\delta} W_{\gamma\delta})|_\beta + \theta K_{\alpha\beta} \overset{0}{v}^{\alpha\beta\gamma\delta} E_{\gamma\delta} + \tfrac{1}{2} P^3 + \tfrac{1}{2}\delta Q^\alpha|_\alpha = O(\varepsilon\gamma^2\delta) \tag{12b}$$

$$\tau^{\alpha 3} = \tfrac{1}{2} Q^\alpha + \tfrac{1}{2}\zeta P^\alpha + \tfrac{1}{2}\delta(1-\zeta^2) M^{\alpha\beta}|_\beta + O(\varepsilon\delta^2 \text{ or } \varepsilon\gamma^2\delta) \tag{13}$$

$$\tau^{33} = \tfrac{1}{2} Q^3 + \tfrac{1}{2}\zeta P^3 + \tfrac{1}{4}\delta(1-\zeta^2) P^\alpha|_\alpha - \tfrac{1}{6}\delta^2\zeta(1-\zeta^2) M^{\alpha\beta}|_{\alpha\beta} + \tfrac{1}{2}\theta(1-\zeta^2) K_{\alpha\beta} M^{\alpha\beta} + O(\varepsilon\gamma^2\delta). \tag{14}$$

The six distinct components of $\overset{0}{v}^{\alpha\beta\gamma\delta}$ are easily expressed in terms of the coefficients of μ_{ijkl} and the equations (11) and (12) are seen to be exactly those of standard treatments. It should be emphasized here, however, that these equations have been obtained without any assumptions of elastic symmetry and are valid for a general anisotropic material provided that E and W are of the same order of magnitude.

In cases where E and W are of different orders of magnitude our equations require further refinement which does lead to the introduction of other components of the elasticity tensor into the basic equations of the theory.

3. Refined Equations

One difference between the anisotropic shell equations (11), (12) and those of an isotropic theory is that the error terms in the present case are somewhat larger than those of the isotropic theory. In the present section the error terms are reduced to the same order of magnitude as the isotropic theory. In the process equations which will apply when E and W are of different orders of magnitude are obtained and it is seen that other components of the elasticity tensor enter the equations.

To achieve greater accuracy it is first necessary to obtain more accurate expressions for $\varepsilon_{\alpha\beta}$. This is done by the use of equations (5), (7), (10) and (13). It is seen that

$$\varepsilon_{\alpha 3} = \overset{0}{\mu}_{\alpha 3\gamma\delta} \overset{0}{v}^{\gamma\delta\sigma\tau} \varepsilon_{\sigma\tau} + O(\varepsilon\delta) = h^{\sigma\tau}_{\alpha 3} E_{\sigma\tau} + \zeta h^{\sigma\tau}_{\alpha 3} W_{\sigma\tau} + O(\varepsilon\delta)$$

where $h^{\sigma\tau}_{\alpha 3} = \overset{0}{\mu}_{\alpha 3\gamma\delta} v^{\gamma\delta\sigma\tau}$ and hence that

$$\varepsilon_{\alpha\beta} = E_{\alpha\beta} + \zeta W_{\alpha\beta} + \delta\zeta(h^{\sigma\tau}_{\alpha 3} E_{\sigma\tau})|_\beta + \delta\zeta(h^{\sigma\tau}_{\beta 3} E_{\sigma\tau})|_\alpha + \tfrac{1}{2}\delta\zeta^2(h^{\sigma\tau}_{\alpha 3} W_{\sigma\tau})|_\beta + \tfrac{1}{2}\delta\zeta^2(h^{\sigma\tau}_{\beta 3} W_{\sigma\tau})|_\alpha + O(\varepsilon\gamma^2).$$

Inversion of equation (4) and the substitution of (13), (14) gives,

$$\tau^{\alpha\beta} = \mathring{v}^{\alpha\beta\gamma\delta} E_{\gamma\delta} + \zeta \mathring{v}^{\alpha\beta\gamma\delta} W_{\gamma\delta} + 2\delta\zeta \mathring{v}^{\alpha\beta\gamma\delta} (h_{\gamma 3}^{\sigma\tau} E_{\sigma\tau})|_{\delta}$$
$$+ \delta\zeta^2 \mathring{v}^{\alpha\beta\gamma\delta} (h_{\gamma 3}^{\sigma\tau} W_{\sigma\tau})|_{\delta} - h_{\sigma 3}^{\alpha\beta} Q^{\sigma} - \zeta h_{\sigma 3}^{\alpha\beta} P^{\sigma} - \delta(1-\zeta^2) h_{\sigma 3}^{\alpha\beta} (v^{\sigma\tau\gamma\delta} W_{\gamma\delta})|_{\tau} + O(\varepsilon\gamma^2) \quad (15a)$$

$$\tau^{\alpha 3} = \tfrac{1}{2}(Q^{\alpha} + \zeta P^{\alpha}) + \tfrac{1}{2}\delta(1-\zeta^2)(\mathring{v}^{\alpha\beta\gamma\delta} W_{\gamma\delta})|_{\beta} + O(\varepsilon\gamma^2) \quad (15b)$$

$$\tau^{33} = O(\varepsilon\gamma^2)\,. \quad (15c)$$

These expressions are substituted into the equilibrium equations (1), (2) which are then integrated with respect to ζ. The boundary conditions are applied at $\zeta = \pm 1$ and the following equations result.

$$\delta(\mathring{v}^{\alpha\beta\gamma\delta} E_{\gamma\delta})|_{\beta} + \tfrac{1}{3}\delta^2 (\mathring{v}^{\alpha\beta\gamma\delta} [h_{\gamma 3}^{\sigma\tau} W_{\sigma\tau}]|_{\delta})|_{\beta}$$
$$- \tfrac{2}{3}\delta^2 (h_{\sigma 3}^{\alpha\beta} [v^{\sigma\tau\gamma\delta} W_{\gamma\delta}]|_{\tau})|_{\beta} + \tfrac{1}{2} P^{\alpha} - \delta(h_{\sigma 3}^{\alpha\beta} Q^{\sigma})|_{\beta} = O(\varepsilon\gamma^2\delta) \quad (16a)$$

$$\tfrac{1}{3}\delta^2 (\mathring{v}^{\alpha\beta\gamma\delta} W_{\gamma\delta})|_{\alpha\beta} + \tfrac{2}{3}\delta^3 (\mathring{v}^{\alpha\beta\gamma\delta} [h_{\gamma 3}^{\sigma\tau} E_{\sigma\tau}]|_{\delta})|_{\alpha\beta} + \theta \mathring{v}^{\alpha\beta\gamma\delta} K_{\alpha\beta} E_{\gamma\delta} + \tfrac{1}{3}\theta\delta \mathring{v}^{\alpha\beta\gamma\delta} K_{\alpha\beta} (h_{\gamma 3}^{\sigma\tau} W_{\sigma\tau})|_{\delta}$$
$$- \tfrac{1}{3}\theta\delta K_{\alpha\beta} h_{\sigma 3}^{\alpha\beta} (\mathring{v}^{\sigma\tau\gamma\delta} W_{\gamma\delta})|_{\tau} + \tfrac{1}{2} P^3 + \tfrac{1}{2}\delta Q^{\alpha}|_{\alpha} + \tfrac{1}{3}\delta^2 (h_{\sigma 3}^{\alpha\beta} P^{\sigma})|_{\alpha\beta} + \theta K_{\alpha\beta} h_{\sigma 3}^{\alpha\beta} Q^{\sigma} = O(\varepsilon\gamma^4) \quad (16b)$$

$$\tau^{\alpha 3} = \tfrac{1}{2}[Q^{\alpha} + \zeta P^{\alpha}] + \tfrac{1}{2}\delta(1-\zeta^2)(\mathring{v}^{\alpha\beta\gamma\delta} W_{\gamma\delta})|_{\beta} + \delta^2 (1-\zeta^2) [\mathring{v}^{\alpha\beta\gamma\delta} (h_{\gamma 3}^{\sigma\tau} E_{\sigma\tau})|_{\delta}]|_{\beta}$$
$$+ \tfrac{1}{2}\delta(1-\zeta^2)(h_{\sigma 3}^{\alpha\beta} P^{\sigma})|_{\beta} + \tfrac{1}{3}\delta^2 \zeta (1-\zeta^2) [\mathring{v}^{\alpha\beta\gamma\delta} (h_{\gamma 3}^{\sigma\tau} W_{\sigma\tau})|_{\delta}$$
$$+ h_{\sigma 3}^{\alpha\beta} (\mathring{v}^{\alpha\tau\gamma\delta} W_{\gamma\delta})|_{\tau}]|_{\beta} + O(\varepsilon\gamma^2\delta)\,.$$

These equations are the refined equilibrium equations. The compatability equations may also be refined. The refinements may be found in Koiter [6] or Westbrook [4]. In Koiter's formulation the compatability equations are independent of the elasticity tensor and are thus exactly the same as the isotropic case. In the present notation which is the same as [4] the term ε_{33} which appears in [4] as $-v(E_{\gamma}^{\gamma} + \zeta W_{\gamma}^{\gamma})$ must be replaced by $\mathring{\mu}_{33\alpha\beta} \mathring{v}^{\alpha\beta\gamma\delta}(E_{\gamma\delta} + \zeta W_{\gamma\delta})$. In the present work however the compatability equations (8), (9) are found to be adequate.

The relationship between the strains $E_{\alpha\beta}$, $W_{\alpha\beta}$ and the displacements u_k may be found from the strain displacement relations. Let v_k be scaled displacements given by

$$u_k = L v_k$$

then the strain displacement relations for ε_{k3} give

$$\frac{1}{\delta} v_{3,\zeta} = \varepsilon_{33} = O(\varepsilon) \quad \text{or} \quad v_3 = V + O(\varepsilon\delta)$$

$$\frac{1}{\delta} v_{\alpha,\zeta} + v_3|_{\alpha} - \frac{2\theta}{\delta} K_{\alpha}^{\beta} v_{\beta} = 2\varepsilon_{\alpha 3} = 2h_{\alpha 3}^{\sigma\tau} E_{\sigma\tau} + 2\zeta h_{\alpha 3}^{\sigma\tau} W_{\sigma\tau} + O(\varepsilon\delta)$$

hence

$$v_2 = V_{\alpha} - \delta\zeta V|_{\alpha} + 2\delta\zeta h_{\alpha 3}^{\sigma\tau} E_{\sigma\tau} + \delta\zeta^2 h_{\alpha 3}^{\sigma\tau} W_{\sigma\tau} + O(\varepsilon\gamma^2)$$

where V_{α} and V are independent of ζ.

$\varepsilon_{\alpha\beta}$ may now be computed from the above expressions for v_{α} and V, this may then be compared with the expression already obtained and it is seen that

$$E_{\alpha\beta} = \tfrac{1}{2}(V_{\alpha}|_{\beta} + V_{\beta}|_{\alpha}) - \frac{\theta}{\delta} K_{\alpha\beta} V + O(\varepsilon\gamma^2)$$
$$W_{\alpha\beta} = -\delta V|_{\alpha\beta} + O(\varepsilon\gamma^2)\,.$$

As one would expect the compatability equations are satisfied identically by the above expressions.

The refined equations (16) are required only when the extensional strains E and the bending strains W are of different orders of magnitude. The following cases are of interest

(A) $E = O(\varepsilon) \quad W = O(\varepsilon\delta) \quad \theta \leq \delta^3\,.$

(B) $E = O(\varepsilon\delta) \quad W = O(\varepsilon)\,.$

Case (A) is an extensional anisotropic shallow shell theory which could occur for example if the displacements are all of the same magnitude. The governing equations are

$$\delta(\mathring{v}^{\alpha\beta\gamma\delta}E_{\gamma\delta})|_\beta+\tfrac{1}{2}P^\alpha+\delta(h^{\alpha\beta}_{\sigma 3}Q^\sigma)|_\beta=O(\varepsilon\gamma^2\delta) \tag{17}$$

$$\tfrac{1}{3}\delta^2(\mathring{v}^{\alpha\beta\gamma\delta}W_{\gamma\delta})|_{\alpha\beta}+\tfrac{2}{3}\delta^3(\mathring{v}^{\alpha\beta\gamma\delta}(h^{\sigma\tau}_{\gamma 3}E_{\sigma\tau})|_\delta)|_{\alpha\beta}+\theta\mathring{v}^{\alpha\beta\gamma\delta}K_{\alpha\beta}E_{\gamma\delta}$$
$$+\tfrac{1}{2}P^3+\tfrac{1}{2}\delta Q^\alpha|_\alpha+\tfrac{1}{3}\delta^2(h^{\alpha\beta}_{\sigma 3}P^\sigma)|_{\alpha\beta}+\theta K_{\alpha\beta}h^{\alpha\beta}_{\sigma 3}Q^\sigma=O(\varepsilon\gamma^4) \tag{18}$$

$$\delta^2 e^{\alpha\beta}e^{\sigma\tau}E_{\alpha\tau}|_{\sigma\beta}=O(\varepsilon\gamma^4) \tag{19a}$$

$$e^{\alpha\beta}W_{\alpha\gamma}|_\beta\qquad=O(\varepsilon\gamma^2)\,. \tag{19b}$$

The difference between these equations and those of standard treatments is the coupling in equation (18) which suggests that bending of order $\varepsilon\delta$ may accompany extension of a shallow shell or plate or that a normal displacement of the same order of magnitude as the tangential displacement may occur in the extension of a shallow shell or plate. An example of this phenomena is given in section four.

Case (B) which is an anisotropic shell bending theory has the governing equations

$$\delta(\mathring{v}^{\alpha\beta\gamma\delta}E_{\gamma\delta})|_\beta+\tfrac{1}{3}\delta^2(\mathring{v}^{\alpha\beta\gamma\delta}[h^{\sigma\tau}_{\gamma 3}W_{\sigma\tau}]|_\delta)|_\beta+\tfrac{2}{3}\delta^2(h^{\alpha\beta}_{\sigma 3}[\mathring{v}^{\sigma\tau\gamma\delta}W_{\gamma\delta}]|_\tau)|_\beta+\tfrac{1}{2}P^\alpha=O(\varepsilon\gamma^2\delta)\ldots \tag{20}$$

$$\tfrac{1}{3}\delta^2(\mathring{v}^{\alpha\beta\gamma\delta}W_{\gamma\delta})|_{\alpha\beta}+\tfrac{1}{2}P^3+\tfrac{1}{2}\delta Q^\alpha|_\alpha=O(\varepsilon\gamma^2\delta) \tag{21}$$

$$e^{\alpha\beta}e^{\sigma\tau}(\delta^2 E_{\alpha\tau}|_{\sigma\beta}-\theta K_{\alpha\tau}W_{\sigma\beta})=O(\varepsilon\gamma^4) \tag{22a}$$

$$e^{\alpha\beta}W_{\alpha\gamma}|_\beta=O(\varepsilon\gamma^2)\,. \tag{22b}$$

The difference here is again the coupling in equation (20) which suggests that extension of order at least $\varepsilon\delta$ may accompany bending in a shell or plate. In this case the tangential displacements would be of order δ^2 times the normal displacement.

The equations (18) and (20) are believed to be new and they indicate the role of the components of the elasticity tensor μ_{ijkl} other than the usual six $\mu_{\alpha\beta\gamma\delta}$ in the anisotropic theory.

4. A Simple Example of Case (A).

To illustrate the consequences of full anisotropy we consider the example of an infinite anisotropic plate with a circular hole at the edge of which a constant normal pressure is applied. It is found that the anisotropy produces a normal displacement of the same magnitude as the radial displacement.

To simplify the calculations we assume that the xz plane (z is normal to the plate) is a plane of elastic symmetry and that the elasticity tensor has the following components referred to the xyz coordinate system.

$$\mu_{1111}=\mu_{2222}=1\,,\quad \mu_{1212}=\tfrac{1}{2}(1+\nu)\,,\quad \mu_{1122}=-\nu$$

and all other distinct $\mu_{\alpha\beta\gamma\delta}$ are zero. Also $\mu_{1311}=A_1$, $\mu_{1322}=A_2$, $\mu_{2312}=B$. Of the remaining components $\mu_{1313}\mu_{2323}\mu_{1333}\mu_{2333}$ do not enter the equations and the rest are zero. It is seen that

$$\mathring{v}^{1111}=\mathring{v}^{2222}=\frac{1}{1-\nu^2}\,,\quad \mathring{v}^{1212}=\frac{1}{2(1+\nu)}\,,\quad \mathring{v}^{1122}=\frac{\nu}{1-\nu^2}\,.$$

The radius of the hole is taken as the typical length and the relevant equations in cartesian coordinates are,

$$N_{\alpha\beta,\beta}=0$$

$$N_{11}=\frac{1}{1-\nu^2}[V_{1,1}+\nu V_{2,2}]\,,\quad N_{22}=\frac{1}{1-\nu^2}[\nu V_{1,1}+V_{2,2}]\,,\quad N_{12}=\frac{1}{2(1+\nu)}[V_{1,2}+V_{2,1}]$$

$$M_{\alpha\beta;\alpha\beta}=0$$

$$M_{11} = -\frac{1}{1-\nu^2}[V_{,11}+\nu V_{,22}] + \frac{2}{1-\nu^2}[A_1 N_{11,1}+A_2 N_{22,1}+\nu B N_{12,2}]$$

$$M_{22} = -\frac{1}{1-\nu^2}[\nu V_{,11}+V_{,22}] + \frac{2}{1-\nu^2}[\nu A_1 N_{11,1}+\nu A_2 N_{22,1}+B N_{12,2}]$$

$$M_{12} = M_{21} = -\frac{1}{(1+\nu)} V_{,12} + \frac{1}{1+\nu}[A_1 N_{11,2}+A_2 N_{22,2}+B N_{12,1}]$$

at infinity the conditions are $N_{\alpha\beta}\to 0$ $M_{\alpha\beta}\to 0$ and at the hole $r=(x^2+y^2)^{\frac{1}{2}}=1$ the boundary conditions are

$$N_{rr} = N_{11}\cos^2\theta+2N_{12}\cos\theta\sin\theta+N_{22}\sin^2\theta = -\frac{P}{E}$$

$$N_{r\theta} = (N_{22}-N_{11})\cos\theta\sin\theta+N_{12}(\cos^2\theta-\sin^2\theta)=0$$

$$M_{rr} = M_{11}\cos^2\theta+2M_{12}\cos\theta\sin\theta+M_{22}\sin^2\theta=0$$

$$S_{r3}+\frac{\partial}{\partial\theta}M_{r\theta}=0\,,\quad S_{r3}=M_{1\beta,\beta}\cos\theta+M_{2\beta,\beta}\sin\theta$$

$$M_{r\theta} = (M_{22}-M_{11})\cos\theta\sin\theta+M_{12}(\cos^2\theta-\sin^2\theta)$$

(S_{r3} is the resultant radial shear stress).

The problem for $N_{\alpha\beta}$ and V_α is uncoupled and is in fact the problem of the isotropic plate with a hole, which has the solution

$$V_r = \frac{P}{E}\frac{(1+\nu)}{r}\,,\quad V_\theta = 0$$

$$N_{rr} = -\frac{P}{Er^2}\,,\quad N_{r\theta}=0\,,\quad N_{\theta\theta} = \frac{P}{Er^2}$$

in polar coordinates. These values may now be inserted into the remaining equations and it is found that the solution is given by

$$V = (A_1-A_2+B)\frac{P}{E}\frac{\cos\theta}{r} + \frac{2(1+\nu)}{(3+\nu)}\frac{(A_2-A_1+B)}{E}P\left[\frac{1}{2r}-\frac{1}{3r^3}\right]\cos 3\theta$$

$$M_{rr} = -\frac{8P}{E}\frac{(A_2-A_1+B)}{(3+\nu)}\left[\frac{1}{r^3}-\frac{1}{r^5}\right]\cos 3\theta$$

$$M_{r\theta} = -\frac{2P}{E}\frac{(A_2-A_1+B)}{(3+\nu)}\left[\frac{3}{r^3}-\frac{4}{r^5}\right]\sin 3\theta$$

$$M_{\theta\theta} = +\frac{2P}{E}\frac{(A_2-A_1+B)}{(3+\nu)}\left[\frac{2}{r^3}-\frac{4}{r^5}\right]\cos 3\theta\,.$$

Thus at the edge of the hole a normal displacement

$$(A_1-A_2+B)\frac{P}{E}\cos\theta + \frac{(1+\nu)(A_2-A_1+B)}{3(3+\nu)}\frac{P}{E}\cos 3\theta$$

is produced by the anisotropy.

5. Conclusion

It is seen that the asymptotic method allows a development of an anisotropic theory which does not require any restrictions on the anisotropy. The use of this method indicates that the equations found in most standard treatments with assumptions on the anisotropy are the same as those found here in many cases, but that under certain circumstances full anisotropy plays a part. It is clear that a theory of multilayered shells or plates with full anisotropy may be developed by the asymptotic method if use is made of the continuity of the stresses τ^{k3} and the strains $\varepsilon_{\alpha\beta}$ (or the displacements) at the interfaces.

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